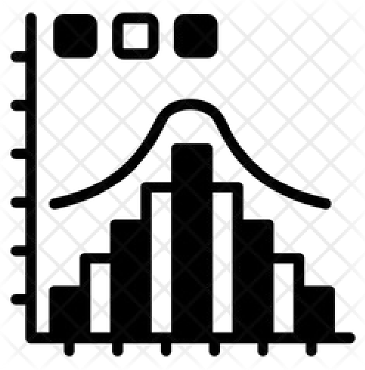
Uniform Distribution



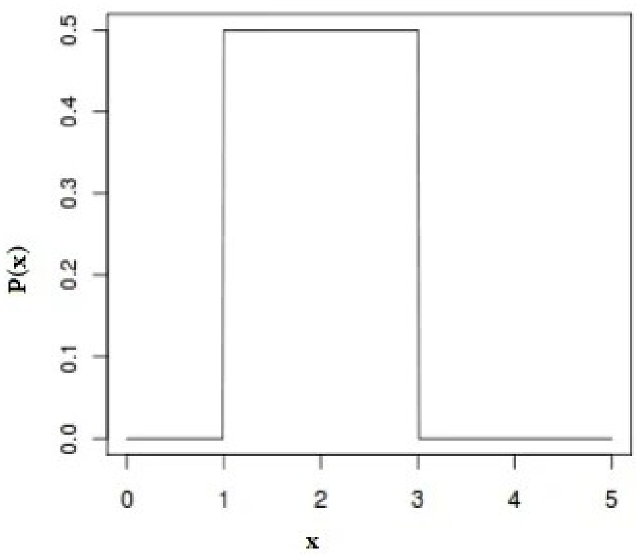
Tushar B. Kute,
http://tusharkute.com

Uniform Distribution

- A uniform distribution, also called a rectangular distribution, is a probability distribution that has constant probability.
- This distribution is defined by two parameters, a and b:
  - a is the minimum.
  - b is the maximum.
- The distribution is written as U(a, b).

Uniform Distribution

- The following graph shows the distribution with a = 1 and b = 3:



Like all probability distributions for continuous random variables, the area under the graph of a random variable is always equal to 1. In the above graph, the area is: A = l x h = 2 * 0.5 = 1.

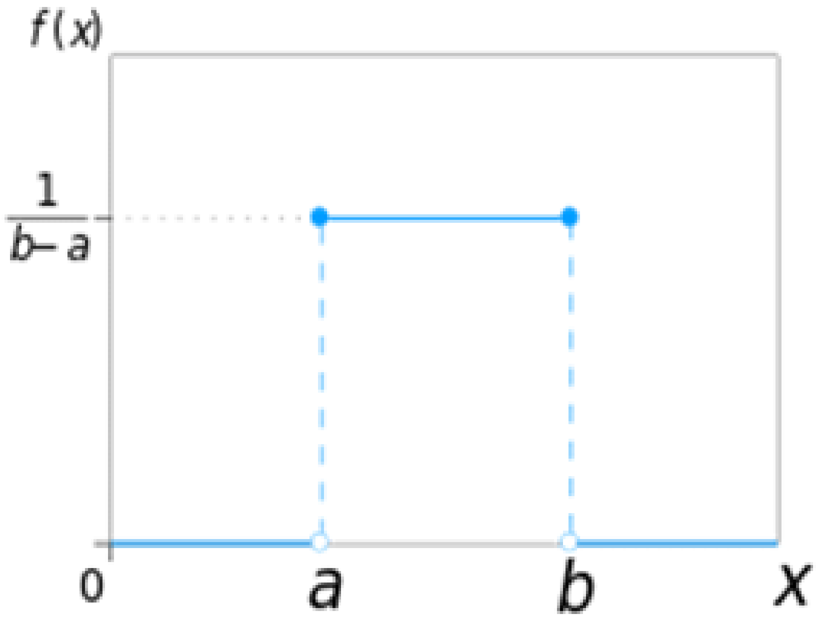
Uniform Distribution – types

- This distribution has two types.
  - The most common type you'll find in elementary statistics is the continuous uniform distribution (in the shape of a rectangle).
  - However, there is a second type: the discrete uniform distribution. It still resembles a rectangle but instead of a line, a series of dots represent a known, finite number of outcomes.

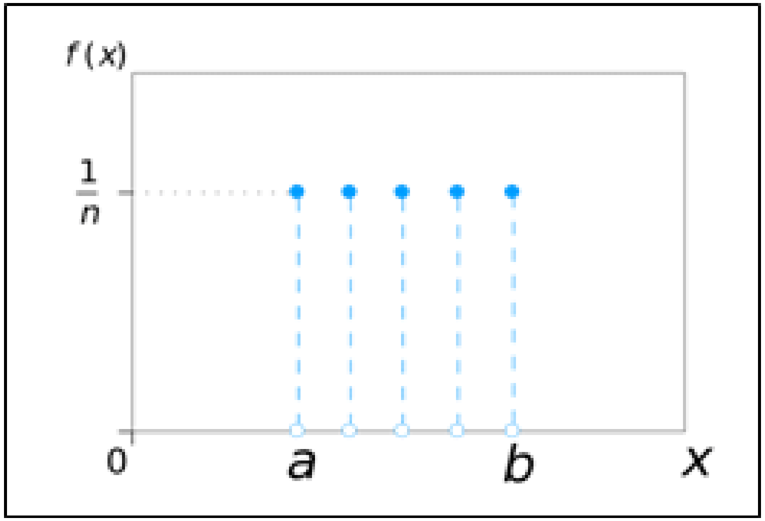
Uniform Distribution – types

Examples of Uniform Distribution			
Probability of landing on each side of a die	Probability of hitting heads or tails	Perfect random number generators	Probability of guessing exact time at any moment
Discrete Uniform Distribution		Continuous Uniform Distribution	

Uniform Distribution – types



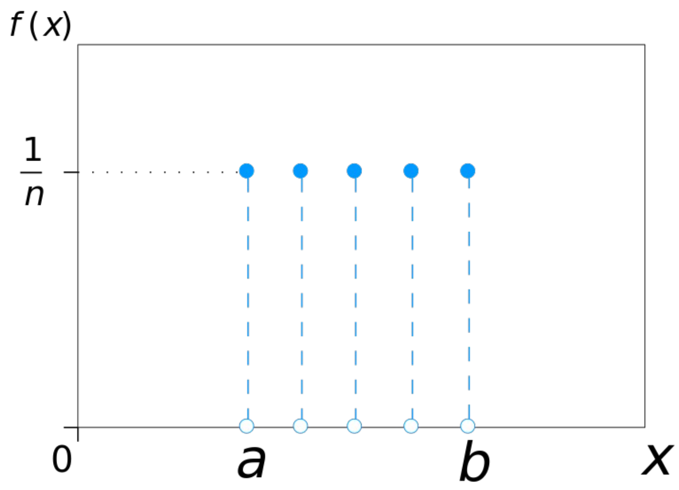
(a) Continuous Uniform Distribution



(b) Discrete Uniform Distribution

Uniform Distribution

- The following graph shows 5 possible outcomes:



Rolling a single die is one example of a discrete uniform distribution; a die roll has four possible outcomes: 1,2,3,4,5, or 6. There is a 1/6 probability for each number being rolled.

General Formula

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

General Formula

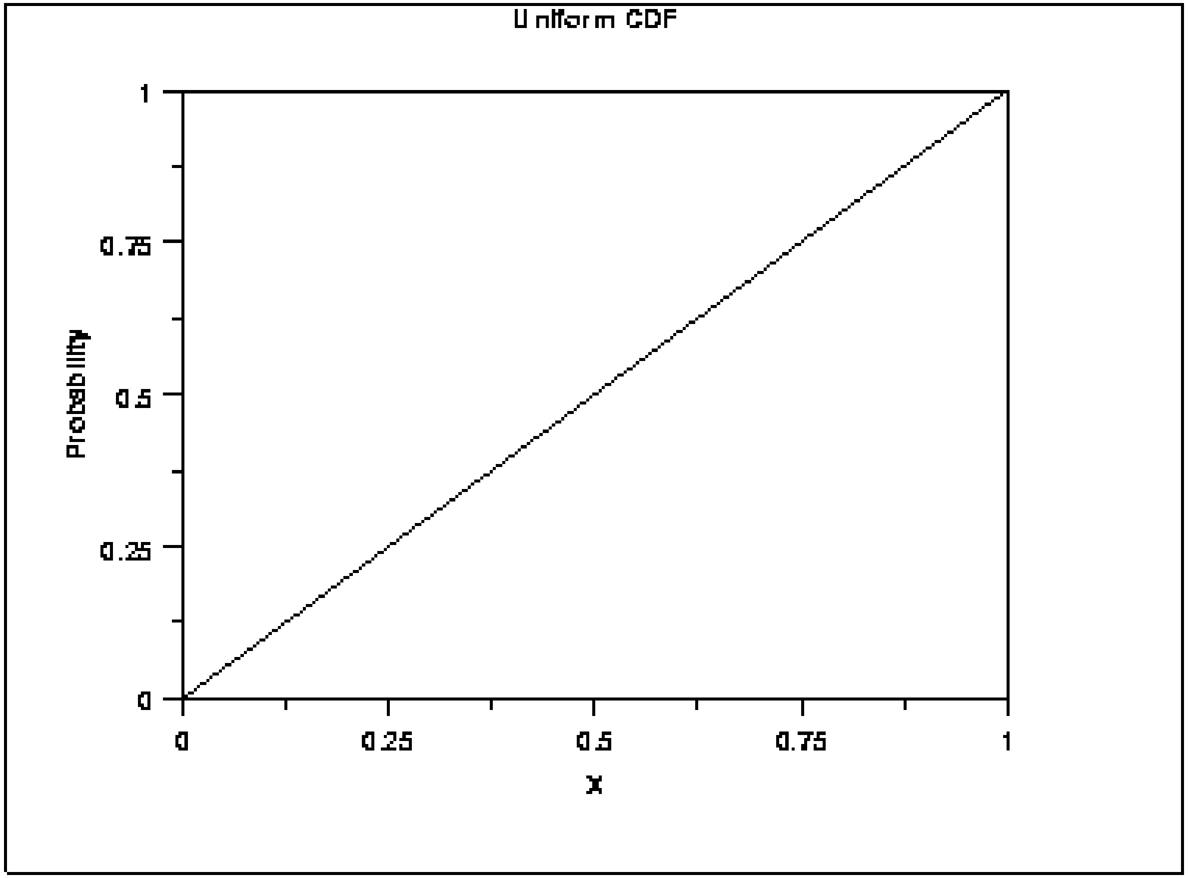
- The general formula for the probability density function (pdf) for the uniform distribution is:

  $f(x) = 1/ (B-A)$ for $A \leq x \leq B$.
- "A" is the location parameter: The location parameter tells you where the center of the graph is.
- "B" is the scale parameter: The scale parameter stretches the graph out on the horizontal axis. Note: The A and B here aren't to be confused with lowercase (a,b), which is an open interval.

Uniform CDF

- The uniform distribution doesn’t always look like a rectangle.
- A special case, the uniform cumulative distribution function, adds up all of the probabilities (in the same way a cumulative frequency distribution adds probabilities) and plots the result, which is a linear graph and not a rectangle:

Uniform CDF



Mean

- The expected value (i.e. the mean) of a uniform random variable X is:

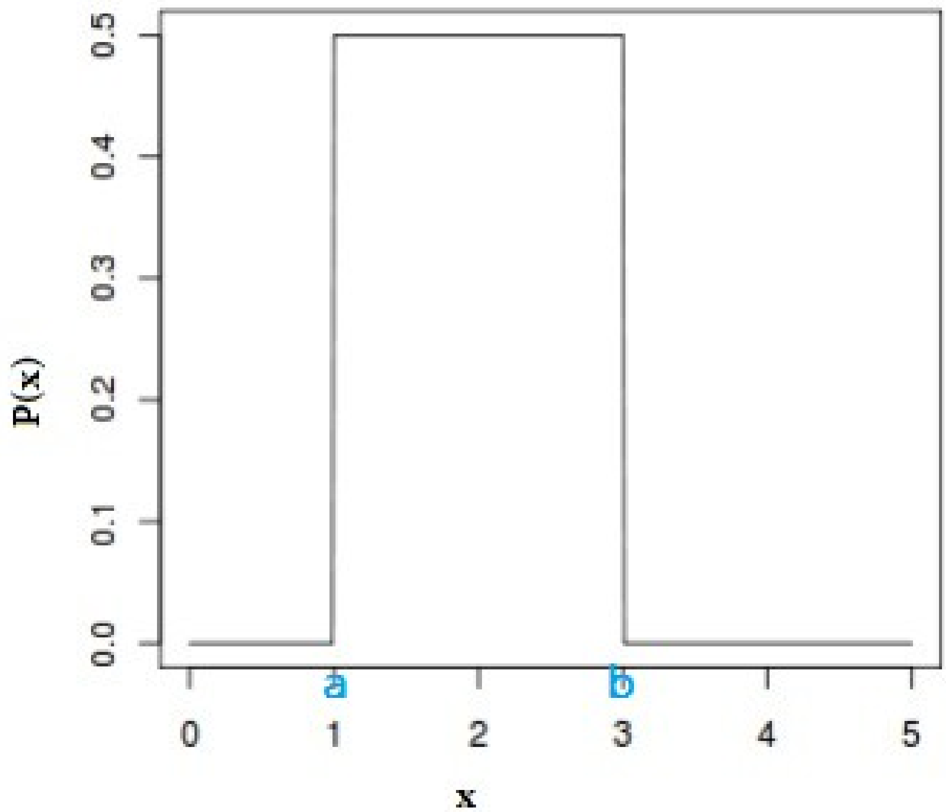
  $$E(X) = (1/2)\,(a + b)$$

- This is also written equivalently as:

  $$E(X) = (b + a) / 2.$$

- “a” in the formula is the minimum value in the distribution, and “b” is the maximum value.

Mean



Mean = (1/2) a + b = 1/2 (1 + 3) = 1/2 (4) = 2

Mean

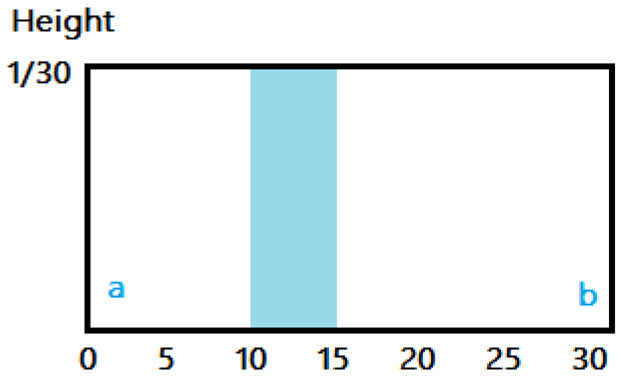
- The variance of a uniform random variable is:

  $Var(x) = (1/12)(b-a)^2$

- For the above image, the variance is, $(1/12)(3 – 1)^2 = 1/12 * 4 = 1/3$.

Finding Probabilities

- Example question #1: The average amount of weight gained by a person over the winter months is uniformly distributed from 0 to 30lbs.
- Find the probability a person will gain between 10 and 15lbs during the winter months.



Area of blue rectangle
= length * height
= 5 * 1/30 = 1/6

Finding Probabilities

- Step 1: Find the height of the distribution. The area under a probability distribution is always 1. As there are 30 units (from zero to 30), then the height is 1/30.
- Step 2: Find the width of the "slice" of the distribution mentioned in the question. Do this by subtracting the biggest number (b) from the smallest (a), to get b – a = 15 – 10 = 5.
- Step 3: Multiply the width (Step 2) by the height (Step 1) to get:
- Probability = 5 * 1/30 = 5/30 = 1/6.

Finding Probabilities

- Example Question 2: Find $P(X \leq 10)$ for the above question.
- This is asking you to find the probability that the random variable X is less than 10. In other words, you want to know the probability a person will gain up to ten pounds.
- Step 1: Find the width of the "box": $b - a = 10 - 0 = 10$.
- Step 2: Multiply the width (Step 1) by the height. We already know the height is 1/30 (from example question 1), so: $10 * 1/30 = 10/30 = 1/3$.

Finding Probabilities

- Example Question 3: Find $P(20 \le X \le 25)$ for the above question. This is asking the probability of a weight gain between 20 and 25 pounds.
- Step 1: Find the width of the "box": $b - a = 25 - 20 = 5$.
- Step 2: Multiply the width (Step 1) by the height. We already know the height is 1/30 (from example question 1), so:

  $5 * 1/30 = 5/30 = 1/6$.

The more formal formula

- You can solve these types of problems using the steps above, or you can us the formula for finding the probability for a continuous uniform distribution:

  P(X) = d – c / b – a.

- This is also sometimes written as:

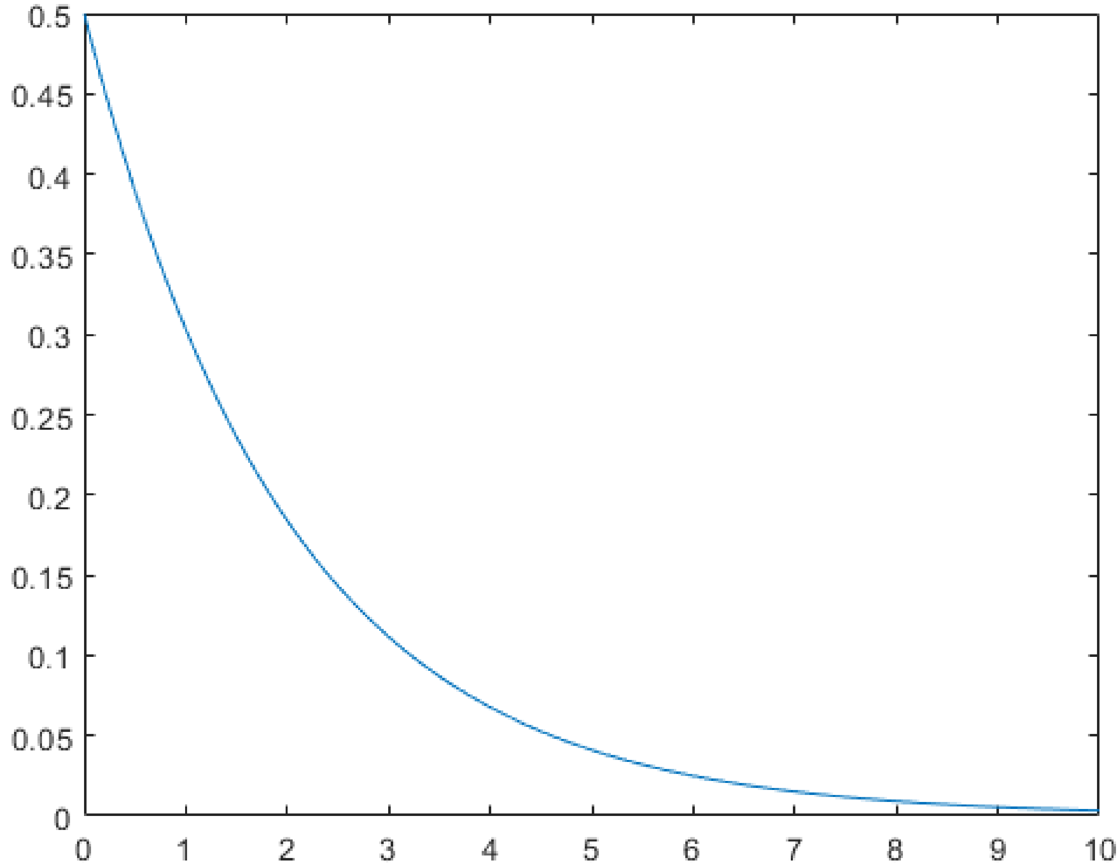
  P(X) = x2 – x1 / b – a.

- “d” and “c” (x2 – x1)are the upper and lower bounds of the area you are trying to find.
- You get exactly the same answer as if you’d followed the steps above. If formulas work for you…great. Personally, I find it easier to visualize these problems as trying to find an area inside a rectangle. Otherwise, I’ve just got another formula to memorize.

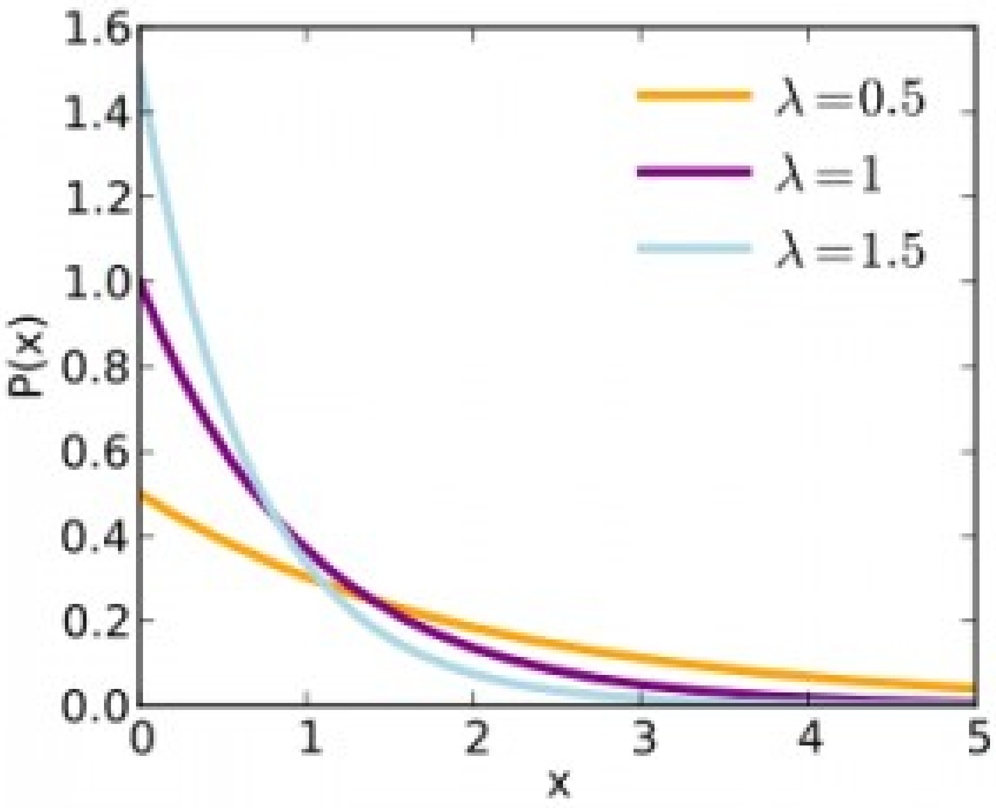
Uniform Distribution in Python

- scipy.stats.uniform() is a Uniform continuous random variable. It is inherited from the of generic methods as an instance of the rv_continuous class. It completes the methods with details specific for this particular distribution.
- Parameters :
  - q : lower and upper tail probability
  - x : quantiles
  - loc : [optional]location parameter. Default = 0
  - scale : [optional]scale parameter. Default = 1
  - size : [tuple of ints, optional] shape or random variates.
  - moments : [optional] composed of letters ['mvsk']; 'm' = mean, 'v' = variance, 's' = Fisher's skew and 'k' = Fisher's kurtosis. (default = 'mv').
  - Results : Uniform continuous random variable

Graph



Probability Density Function



Probability Density Function

- The most common form of the pdf is:

  $F(x;\lambda) = e^{-\lambda x}$ x > 0.
- Where:
  - e = the natural number e,
  - λ = mean time between events,
  - x = a random variable.
- For x less than 0, F(x;λ) = 0

Probability Density Function

- For x less than 0, $F(x;\lambda) = 0$
- An alternate form of the pdf (Engineering Statistics Handbook) is:
- $f(x) = (1/\beta)\, e^{-(x-\mu)/\beta}$ $x \geq \mu$; $\beta > 0$

  where μ is the location parameter and β is the scale parameter. The scale parameter is sometimes referred to as λ which equals 1/β.
- If $\mu = 0$ and $\beta = 1$ it is called the standard exponential distribution and has the equation:

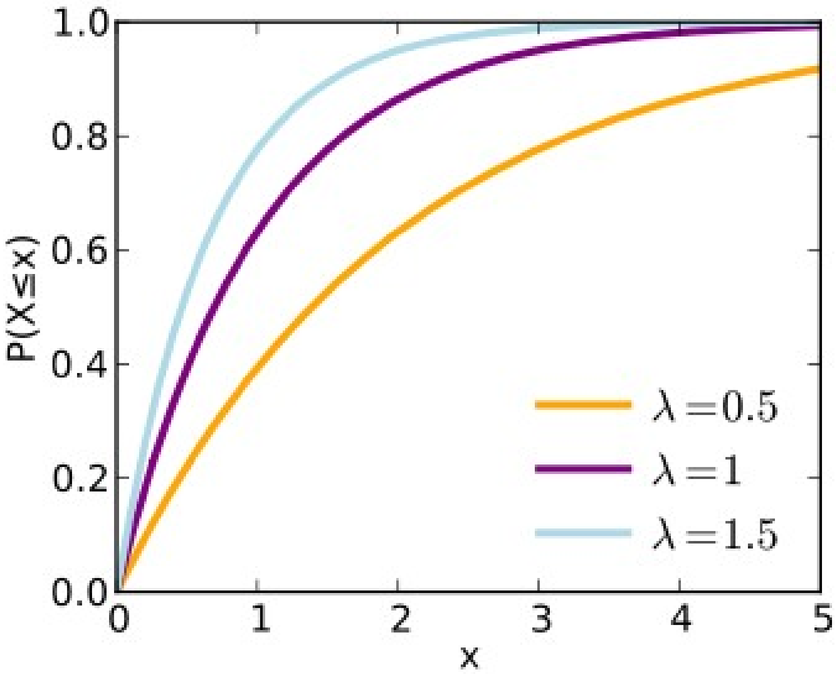
  $f(x)=e^{-x}$ for $x \geq 0$.

Cumulative Distribution Function

- The formula for the cumulative distribution function of the exponential distribution is:

$F(x) = 1 - e^{-\lambda x}$

$x \geq 0; \lambda > 0$



Connection to Other Distributions

- The exponential distribution is related to the Poisson distribution.
- While the exponential models time between successive events over a continuous time interval, the Poisson distribution deals with events that happen over a fixed period of time.
- The number of times the event will happen within a certain amount of time is a Poisson distribution if:
  - The event can happen more than one time.
  - If the time elapsed between two events is exponentially distributed.
  - The events are independent of previous events.

Using Python

- You can generate an exponentially distributed random variable using scipy.stats module's expon.rvs() method which takes shape parameter scale as its argument which is nothing but 1/lambda in the equation.
- To shift distribution use the loc argument, size decides the number of random variates in the distribution.
- If you want to maintain reproducibility, include a random_state argument assigned to a number.

Thank you

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https://mitu.co.in
http://tusharkute.com

contact@mitu.co.in

tushar@tusharkute.com