Valid Arguments and Proof Systems

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Argument

- Definition of Argument:
  - An argument is a sequence of statements in which the conjunction of the initial statements (called the premises/hypotheses) is said to imply the final statement (called the conclusion). An argument can be presented symbolically as:
  - $(P_1 \land P_2 \land \ldots \land P_n) \rightarrow Q$
  - Where $P_1, P_2, \ldots, P_n$ represent the hypotheses and Q represents the conclusion
  - The question can be stated as:
    - when can Q be logically deduced from $P_1, P_2, \ldots, P_n$?
    - when is Q a logical conclusion from $P_1, P_2, \ldots, P_n$?

Argument

- Note: we need to focus on the relationship of the conclusion to the hypotheses and not just any knowledge we might have about the conclusion Q.

- For example:
  - P_1: Neil Armstrong was the first to step on the moon.
  - P_2 : Mars is a red planet.
  - and the conclusion
  - No human has ever been to Mars.

- A valid argument should be true based entirely on its internal structure.

Valid Argument

- Definition of valid argument:
  - An argument is valid if whenever the hypotheses are all true, the conclusion must also be true.
  - That is, when $(P_1 \land P_2 \land \ldots \land P_n) \rightarrow Q$ is a tautology.
  - The previous example had a wff representation of $A \land B \rightarrow C$ which is not a tautology
- Example:
  - If George Bush is the current president of the US, then Dick Cheney is the current vice president. George Bush is not the current president of the US. Therefore Dick Cheney is not the current vice president.
  - $(A \rightarrow B) \land A' \rightarrow B'$

Proof Sequence

- To test whether $(P_1 \land P_2 \land \ldots \land P_n) \rightarrow Q$ is a tautology:
  - build a truth table
  - generate a proof sequence (new way) by applying derivation rules

- Definition of Proof Sequence:
  - A sequence of wffs in which each wff is either a hypothesis or the result of applying one of the formal system's derivation rules to earlier wffs in the sequence
  - The above proof sequence results in many numbers of wffs and finally it will result in the conclusion

Derivation Rules

- Formal logic system that is:
  - correct: only valid arguments should be provable
  - complete: every valid argument should be provable
  - minimum: to make the formal system manageable

- Derivation rules for Propositional Logic
  - Equivalence rules: allows individual wffs to be replaced
  - Inference rules: allows new wffs to be derived from previous wffs

Equivalence Rules

- These rules state that certain pairs of wffs are equivalent, hence one can be substituted for the other with no change to its truth values.
- Allows substitution in either direction

Expression	Equivalent to	Name/Abbreviation
$R \vee S$ $R \wedge S$	$S \vee R$ $S \wedge R$	communicative / **comm**
$(R \vee S) \vee Q$ $(R \wedge S) \wedge Q$	$R \vee (S \vee Q)$ $R \wedge (S \wedge Q)$	associative / **ass**
$(R \vee S)'$ $(R \wedge S)'$	$R' \wedge S'$ $R' \vee S'$	De Morgan's laws / **De Morgan**
$R \rightarrow S$	$R' \vee S$	implication / **imp**
R	$(R')'$	double negation / **dn**
$R \leftrightarrow S$	$(R \rightarrow S) \wedge (S \rightarrow R)$	equivalence / **equ**

Examples

- Assume we have the following hypotheses, we can start a proof sequence as follows:
  1. $(A' \lor B') \lor C$ hyp (hypothesis)
  2. $(A \land B)' \lor C$ 1, De Morgan
  3. $(A \land B) \to C$ 2, imp

Inference Rules

- Inference rules allow us to add a wff to match the last part of the proof sequence, if one or more wffs that match the first part already exist in the proof sequence

From	Can Derive	Abbreviation for rule
R, R → S	S	Modus Ponens- **mp**
R → S, S′	R′	Modus Tollens- **mt**
R, S	R ∧ S	Conjunction-**con**
R ∧ S	R, S	Simplification- **sim**
R	R ∨ S	Addition- **add**

- Note: Inference rules *do NOT* work in both directions unlike equivalence rules
- Example:
  - R: It's bright and sunny today. S: I'll wear my sunglass.
  - mp, mt

Proving Valid Statement

- First, write down all the hypotheses
- Then use the inference and equivalence rules to get to the conclusion step by step
- The idea is to keep focused on the result and sometimes it is very easy to go down a longer path

Deduction Method

- To prove an argument of the form:
  - $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow (R \rightarrow Q)$
- Deduction method allows for the use of R as an additional hypothesis and prove:
  - $P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge R \rightarrow Q$
- Example: prove $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$
- Example, prove $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
- The above is called rule of Hypothetical Syllogism or hs in short
- Many such other rules can be derived from existing rules which thus provide an easier and faster proofs

More Inference Rules

From	Can Derive	Name / Abbreviation
$P \to Q, Q \to R$	$P \to R$	Hypothetical syllogism- hs
$P \vee Q, P'$	Q	Disjunctive syllogism- ds
$P \to Q$	$Q' \to P'$	Contraposition- cont
$Q' \to P'$	$P \to Q$	Contraposition- cont
P	$P \wedge P$	Self-reference - self
$P \vee P$	P	Self-reference - self
$(P \wedge Q) \to R$	$P \to (Q \to R)$	Exportation - exp
P, P'	Q	Inconsistency - inc
$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$	Distributive - dist
$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$	Distributive - dist

Proof of Inference Rules

- Prove that $(P \to Q) \to (Q' \to P')$ is a valid argument (Contraposition – con)
- Hence prove, $(P \to Q) \land Q' \to P'$ (using deduction method)
- The above is true using the modus tollens inference rule
- Prove $P \land P' \to Q$ (Inconsistency -- inc)

1.	P	hyp
2.	P'	hyp
3.	$P \lor Q$	1, add
4.	$Q \lor P$	3, comm
5.	$(Q')' \lor P$	4, dn
6.	$Q' \to P$	5, imp
7.	$(Q')'$	2, 6, mt
8.	Q	7, dn

Proving Verbal Arguments

- An argument in English that consists of simple statements can be tested for validity by a two-step process:
  - Symbolize the argument using propositional wffs
  - Prove that the argument is valid by constructing a proof sequence for it using the derivation rules for propositional logic

Proving Verbal Arguments Ex.

- Russia was a superior power, and either France was not strong or Napoleon made an error. Napoleon did not make an error, but if the army did not fail, then France was strong. Hence the army failed and Russia was a superior power.

- Converting it to a propostional form using letters A, B, C and D
  - A : Russia was a superior power
  - B: France was strong
  - B′ : France was not strong
  - C: Napoleon made an error
  - C′ : Napoleon did not make an error
  - D: The army failed
  - D′ : The army did not fail

Proving Verbal Arguments Ex.

- Combining, the statements using logic
  - $A \land (B' \lor C)$ hypothesis
  - C' hypothesis
  - $(D' \to B)$ hypothesis
  - $(D \land A)$ conclusion

- Combining them, the propositional form is
- $A \land (B' \lor C) \land C' \land (D' \to B) \to (D \land A)$
- Prove it

Thank you

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